

# Anomaly and the self-energy of electric charges

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We study the self-energy of a charged particle located in a static  $D$ -dimensional gravitational field. We show that the energy functional for this problem is invariant under an infinite dimensional (gauge) group of transformations parametrized by one scalar function of  $(D - 1)$  variables. We demonstrate that the problem of the calculation of the self-energy for a pointlike charge is equivalent to the calculation of the fluctuations  $\langle \psi^2 \rangle$  for an effective  $(D - 1)$ -dimensional Euclidean quantum field theory. Using point-splitting regularization we obtain an expression for the self-energy and show that it possesses anomalies. Explicit calculation of the self-energy and its anomaly is done for the higher dimensional Majumdar-Papapetrou spacetimes.

## I. INTRODUCTION

Recently it was demonstrated that the problem of the self-energy of a pointlike scalar charge in a  $D$ -dimensional static gravitational field can be reduced to the problem of calculation of vacuum fluctuations  $\langle \varphi^2 \rangle$  of a scalar field in the effective  $(D - 1)$ -Euclidean quantum field theory. This theory, besides a dynamical field  $\varphi$  includes  $(D - 1)$ -dimensional metric  $g_{ab}$  and a dilaton field  $\alpha$ , which is related to the  $g_{tt}$  component of the metric of the original theory. Moreover, the energy, which plays the role of the  $(D - 1)$ -dimensional Euclidean action, possesses the property of gauge invariance with respect to joint transformation of  $\varphi$ ,  $g_{ab}$ , and  $\alpha$ . Standard regularizations, required to make the self-energy finite, break this invariance. As a result, the renormalized expression for the self-energy and the mass shift for a scalar pointlike charge contains an anomaly. This anomalous contribution vanishes for even  $D$  and is nontrivial in odd dimensional spacetimes. This anomaly was calculated and discussed in [1].

In this paper we demonstrate that a similar anomaly exists for the self-energy of static pointlike electric charges in the higher dimensional Maxwell theory. In a static  $D$ -dimensional spacetime, this problem also can be reduced to the calculation of the renormalized value of the fluctuations  $\langle \psi^2 \rangle$  for some effective  $(D - 1)$ -dimensional Euclidean quantum scalar field  $\psi$ . The effective  $(D - 1)$ -dimensional action for this theory again is gauge invariant for special transformation of the field and background variables. We calculate the corresponding anomaly, and apply the obtained results for the calculation of the electromagnetic mass shift in some special static spacetimes, containing black holes.

## II. SELF-ENERGY OF AN ELECTRIC CHARGE IN STATIC SPACETIMES

Let us consider an electric charge  $e$  in a static  $D$ -dimensional spacetime with the metric  $g_{\mu\nu}$

$$ds^2 = -\alpha^2 dt^2 + g_{ab} dx^a dx^b. \quad (2.1)$$

We assume that the spacetime is static  $\partial_t \alpha = \partial_t g_{ab} = 0$ . The action for the Maxwell field in higher dimensions is

$$I = -\frac{1}{16\pi} \int d^D y \sqrt{-g} F^{\mu\nu} F_{\mu\nu} + \int d^D y \sqrt{-g} A_\mu J^\mu. \quad (2.2)$$

Here  $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$  and

$$g = \det g_{\mu\nu} = -\alpha^2 g, \quad g = \det g_{ab}. \quad (2.3)$$

The Greek indices  $\alpha, \beta, \dots$  mark spacetime coordinates, while Latin indices  $a, b, \dots$  correspond to spatial coordinates.

The field obeys the equation

$$F^{\mu\epsilon}{}_{;\epsilon} = 4\pi J^\mu. \quad (2.4)$$

The electromagnetic stress-energy tensor is

$$T_{\mu\nu} = \frac{1}{4\pi} \left( F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right). \quad (2.5)$$

For a static source  $J^\mu = \delta_0^\mu J^0$  the only nonvanishing components of the Maxwell tensor are  $F_{0a} = -F_{a0}$ . In the Coulomb gauge the Maxwell tensor reads  $F_{0a} = -\partial_a A_0$ . The nontrivial Maxwell equations

$$F^{0\epsilon}{}_{;\epsilon} = 4\pi J^0, \quad (2.6)$$

when rewritten in terms of the vector potential  $A_\alpha = (A_0, A_a)$ , are

$$\frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} g^{00} g^{\alpha\beta} \partial_\beta A_0) = -4\pi J^0. \quad (2.7)$$

Without loss of generality, one can put  $A_a = 0$  and fix the asymptotic value of the potential at spatial infinity  $A_0|_\infty = 0$ .

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The energy  $E$  of a static configuration of fields is

$$\begin{aligned} E &= -\frac{1}{8\pi} \int_{\mathcal{M}} d^{D-1}x \sqrt{-g} F^{0a} F_{0a} \\ &= \frac{1}{8\pi} \int d^{D-1}x \sqrt{g} \alpha^{-1} g^{ab} \partial_a A_0 \partial_b A_0. \end{aligned} \quad (2.8)$$

In the presence of black holes the integration is over the space  $\mathcal{M}$  lying outside the horizons. By integration by parts we rewrite this spatial integral as the bulk integral and a set of  $(D-2)$ -dimensional surface integrals over the black hole horizons  $\Sigma_k$  and boundary at spatial infinity  $\Sigma_\infty$ .

$$\begin{aligned} E &= -\frac{1}{2} \int_{\mathcal{M}} d^{D-1}x \sqrt{-g} A_0 J^0 \\ &\quad - \frac{1}{8\pi} \int_{\Sigma_\infty} d\sigma_a \sqrt{-g} g^{00} g^{ab} A_0 \partial_b A_0 \\ &\quad - \frac{1}{8\pi} \int_{\Sigma_k} d\sigma_a \sqrt{-g} g^{00} g^{ab} A_0 \partial_b A_0. \end{aligned} \quad (2.9)$$

We choose  $A_0|_{\Sigma_\infty} = 0$  at the boundary at infinity. The boundary integral at infinity is proportional to the charge of the source and to  $A_0|_{\Sigma_\infty}$ . Hence it vanishes.

The generic property of black holes is that the vector potential  $A_0$  on every horizon is constant. Therefore the surface integrals over every horizon are proportional to these constants and to the total flux across the horizon of the electric field created by the source. Because the charge is located outside the horizons this flux is identically zero. Thus all surface integrals vanish. The bulk integral can be represented in terms of the static Green function  $\mathcal{G}_{00}$  of the Maxwell field

$$A_0 = 4\pi \int_{\mathcal{M}} d^{D-1}x' \sqrt{-g(x')} \mathcal{G}_{00}(x, x') J^0(x') \quad (2.10)$$

where  $\mathcal{G}_{00}$  is the solution of the problem

$$\frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} g^{00} g^{ab} \partial_b) \mathcal{G}_{00} = -\frac{\delta^{D-1}(x-x')}{\sqrt{-g}}. \quad (2.11)$$

Eventually we get

$$\begin{aligned} E &= -2\pi \int d^{D-1}x d^{D-1}x' \sqrt{-g(x)} \sqrt{-g(x')} \\ &\quad J^0(x) \mathcal{G}_{00}(x, x') J^0(x'). \end{aligned} \quad (2.12)$$

It is convenient to introduce another field variable  $\psi$  instead of the electric potential

$$A_0 = -\alpha^{1/2} \psi. \quad (2.13)$$

Then we can rewrite our problem as that for the scalar field  $\psi$  in  $(D-1)$ -dimensional space and in the presence of the external dilaton field  $\alpha$ .

The equation for the field  $\psi$  can be derived from Eq.(2.11)

$$\mathcal{O} \psi = -4\pi j, \quad \mathcal{O} \equiv (\Delta + V). \quad (2.14)$$

Here,

$$\Delta = g^{ab} \nabla_a \nabla_b \quad (2.15)$$

is the  $(D-1)$ -dimensional covariant Laplace operator,  $V$  is the potential, and  $j$  is the effective scalar charge density

$$\begin{aligned} V &= -\frac{3}{4} \frac{(\nabla \alpha)^2}{\alpha^2} + \frac{\Delta \alpha}{2\alpha} \equiv -\alpha^{1/2} \Delta(\alpha^{-1/2}), \\ j &\equiv \alpha^{3/2} J^0. \end{aligned} \quad (2.16)$$

The field  $\psi$  is chosen in such a way that the operator  $\mathcal{O}$  is self-adjoint in the space with the metric  $g_{ab}$ .

In terms of the field  $\psi$  the energy Eq.(2.8) takes the form

$$E = \frac{1}{8\pi} \int d^{D-1}x \sqrt{g} g^{ab} \left( \psi_{,a} + \frac{\alpha_{,a}}{2\alpha} \psi \right) \left( \psi_{,b} + \frac{\alpha_{,b}}{2\alpha} \psi \right), \quad (2.17)$$

or, taking into account Eq.(2.12), one can write

$$\begin{aligned} E &= -2\pi \int d^{D-1}x d^{D-1}x' \sqrt{g(x)} \sqrt{g(x')} \\ &\quad j(x) \mathcal{G}(x, x') j(x'). \end{aligned} \quad (2.18)$$

Here  $\mathcal{G}(x, x')$  is the Green function, corresponding to the operator  $\mathcal{O} = \Delta + V$ ,

$$(\Delta + V) \mathcal{G}(x, x') = -\delta^{D-1}(x, x') \quad (2.19)$$

The Green functions  $\mathcal{G}$  and  $\mathcal{G}_{00}$  are related to each other as follows

$$\mathcal{G}_{00}(x, x') = -\alpha^{1/2}(x) \alpha^{1/2}(x') \mathcal{G}(x, x'). \quad (2.20)$$

### III. SYMMETRY PROPERTY OF THE SELF-ENERGY

Consider the following transformations of the metric Eq.(2.1) and of the field  $\psi$

$$g_{ab} = \Omega^2 \bar{g}_{ab}, \quad \alpha = \Omega^n \bar{\alpha}, \quad \psi = \Omega^{-n/2} \bar{\psi}, \quad (3.1)$$

where  $n \equiv D-3$ . For these transformations one has

$$A_0 = \bar{A}_0. \quad (3.2)$$

From the point of view of a field theory on a  $(D-1)$ -dimensional spatial slice, Eq.(3.1) describe simultaneous conformal transformation of the metric  $g_{ab}$  and transformation of the dilaton field  $\alpha$ . Under these transformations the energy functional Eq.(2.8) remains invariant.

The operator  $\mathcal{O}$  in Eq.(2.14) transforms homogeneously

$$\mathcal{O} = \Omega^{-2-\frac{n}{2}} \bar{\mathcal{O}} \Omega^{\frac{n}{2}}, \quad (3.3)$$

That is

$$(\Delta + V)\psi = \Omega^{-2-\frac{n}{2}}(\bar{\Delta} + \bar{V})\bar{\psi}, \quad (3.4)$$

It should be noted that these symmetry transformations Eq.(3.1) differ from those for the self-energy of a scalar charge [1]. In the scalar case the transformation of the dilaton field  $\alpha$  has different power of the conformal factor.

The energy  $E$  Eq.(2.8) is a functional of  $(D-1)$ -dimensional dynamical field  $\psi$  and two external fields  $g_{ab}$  and  $\alpha$ . The transformations Eq.(3.1) preserve the value of this functional. Our effective  $(D-1)$ -dimensional Euclidean field theory happens to be invariant under infinite-dimensional group parametrized by one function  $\Omega(x)$ . This property is similar to the conformal symmetry. It becomes the conformal invariance of the theory only in four dimensions. Let us note that the Eq.(2.4) for the Maxwell field  $A_\mu$  in spacetimes with the dimension  $D > 4$  is not conformally invariant.

#### IV. CLASSICAL ANOMALY

The invariance with respect to the transformations Eq.(3.1) describes the classical symmetry of the system. For a pointlike charge, the classical functional Eq.(2.8) diverges. The divergent part of the electromagnetic energy of a charge can be recombined with the contribution of nonelectromagnetic fields, which are responsible for the stability of the charge and also contribute to its bare mass. After this renormalization one obtains the finite total mass of the charge. But in a general case nonelectromagnetic fields do not respect the observed symmetry. This is the cause of an anomalous contribution to the self-energy of charges in curved spacetimes. In the limit of a pointlike charge, the details of the structure of the classical model of the charge become unimportant and can be described by a universal function. The regularization methods of quantum field theory provide us with the proper tools to deal with the divergencies. In quantum field theory the fact that renormalization procedure breaks some symmetries of the classical theory is the cause of appearance of conformal, chiral and other anomalies. In our case the same arguments are applicable to the renormalized self-energy of classical sources. For the same reason their self-energy acquires anomalous terms. All traditional methods of UV regularization like point-splitting, zeta-function and dimensional regularizations, proper time cutoff, Pauli-Villars and other approaches are applicable to the calculation of the self-energy. For our problem the most natural choice is to use the point-splitting regularization.

The self-energy of a scalar charge has been studied in our previous papers [1, 2]. We have shown that in a generic case it can be written as a sum of the self-energy in some reference spacetime and the anomalous term. The anomaly proves to vanish in even-dimensional

spacetimes, while is nontrivial in odd-dimensional spacetimes.

The case of an electric charge can be treated along the same lines, in spite of the fact that the symmetry transformations Eq.(3.1) are different from those of the scalar sources. The renormalized self-energy Eq.(2.18) of a pointlike charge  $e$  described by the current

$$J^0 = e \alpha^{-1} \delta^{n+2}(x, x') \quad (4.1)$$

takes the form [8]

$$E_{\text{ren}} \equiv \alpha \Delta m, \quad (4.2)$$

$$\Delta m = 2\pi e^2 \mathcal{G}_{\text{reg}}(x, x). \quad (4.3)$$

Here  $\mathcal{G}_{\text{reg}}(x, x)$  is the coincidence limit  $x' \rightarrow x$  of the regularized Green function

$$\mathcal{G}_{\text{reg}}(x, x') = \mathcal{G}(x, x') - \mathcal{G}_{\text{div}}(x, x'). \quad (4.4)$$

The Green function  $\mathcal{G}$  corresponds to the operator Eq.(2.14). Thus, in order to find out the self-energy of an electric charge one has to know the regularized Euclidean Green functions  $\mathcal{G}_{00\text{reg}}(x, x)$  or  $\mathcal{G}_{\text{reg}}(x, x)$ . The latter one in the limit of coincident points is exactly the  $\langle \psi^2 \rangle_{\text{ren}}$  of the scalar field  $\psi$ . In other words technically the problem of calculation of  $\Delta m$  is formally equivalent to computation of the quantum vacuum average value of  $\langle \psi^2 \rangle_{\text{ren}}$  in  $(D-1)$ -dimensional space.

Similar to the scalar case, [1] the Green function Eq.(2.19) transforms as follows

$$\mathcal{G}(x, x') = \Omega^{-\frac{n}{2}}(x) \bar{\mathcal{G}}(x, x') \Omega^{-\frac{n}{2}}(x'). \quad (4.5)$$

Formally the nonrenormalized value of  $\langle \psi^2 \rangle$  transforms homogeneously

$$\langle \psi^2(x) \rangle = \Omega^{-n}(x) \langle \bar{\psi}^2(x) \rangle. \quad (4.6)$$

Thus the combination

$$g^{\frac{n}{2(n+2)}} \langle \psi^2 \rangle \quad (4.7)$$

is formally invariant under the transformations Eq.(3.1). The regularization procedure breaks this classical symmetry and  $g^{\frac{n}{2(n+2)}} \langle \psi^2 \rangle_{\text{ren}}$  is not invariant anymore. However, one can find such finite term  $\mathcal{A}(x)$  that makes the combination

$$g^{\frac{n}{2(n+2)}} (\langle \psi^2 \rangle_{\text{ren}} + \mathcal{A}) \quad (4.8)$$

invariant.

Following exactly to the lines of the paper [1] one can show that

$$\langle \psi^2 \rangle_{\text{ren}} = \Omega^{-n} \langle \bar{\psi}^2 \rangle_{\text{ren}} - \mathcal{B}, \quad (4.9)$$

where

$$\mathcal{B}(x) = \mathcal{A}(x) - \Omega^{-n} \bar{\mathcal{A}}(x) \quad (4.10)$$

is defined as

$$\mathcal{B}(x) = \lim_{x' \rightarrow x} \left[ \mathcal{G}_{\text{div}}(x, x') - \frac{\bar{\mathcal{G}}_{\text{div}}(x, x')}{\Omega^{n/2}(x) \Omega^{n/2}(x')} \right]. \quad (4.11)$$

In order to find the explicit form of this finite anomalous term  $\mathcal{B}$  one can apply the Hadamard representation of the Green functions. The divergent part of the Green function [2] reads

$$\begin{aligned} \mathcal{G}_{\text{div}}(x, x') &= \Delta^{1/2}(x, x') \frac{1}{(2\pi)^{\frac{n}{2}+1}} \\ &\times \sum_{k=0}^{[n/2]} \frac{\Gamma(\frac{n}{2} - k)}{2^{k+1} \sigma^{\frac{n}{2}-k}} a_k(x, x'). \end{aligned} \quad (4.12)$$

For even  $n$  the last term ( $k = n/2$ ) in the sum is to be replaced by

$$\begin{aligned} &\frac{\Gamma(\frac{n}{2} - k)}{2^{k+1} \sigma^{\frac{n}{2}-k}} a_k(x, x') \Big|_{k=n/2} \\ &\rightarrow -\frac{\ln \sigma(x, x') + \gamma - \ln 2}{2^{\frac{n}{2}+1}} a_{n/2}(x, x'). \end{aligned} \quad (4.13)$$

Here  $a_k(x, x')$  are the Schwinger–DeWitt coefficients for the operator  $O$ . The world function  $\sigma(x, x')$  and Van Vleck–Morette determinant  $\Delta(x, x')$  are defined on the  $(n+2)$ -dimensional space with the metric  $g_{ab}$ .

Thus, the self-energy of an electric charge takes the form  $E_{\text{ren}} \equiv \alpha \Delta m$ , where

$$\begin{aligned} \Delta m &= 2\pi e^2 \langle \psi^2 \rangle_{\text{ren}} \\ &= 2\pi e^2 [\Omega^{-n} \langle \bar{\psi}^2 \rangle_{\text{ren}} - \mathcal{B}]. \end{aligned} \quad (4.14)$$

The first term in brackets describes the proper mass calculated in a reference spacetime  $\bar{g}_{\mu\nu}$ . This term takes into account the dependence of the self-energy on boundary conditions and other IR properties of the electromagnetic field in curved spacetime. The second term comes out of the anomaly in question and reflects UV behavior of the system.

There are physically interesting spacetimes where the self-energy can be computed exactly. At any rate the anomalous contribution is local and it can be computed exactly in any static spacetime. In several special cases the reference space can be chosen in such a way that the calculation of vacuum fluctuations  $\langle \bar{\psi}^2 \rangle_{\text{ren}}$  becomes simple or exactly solvable.

The scalar and electric charges near the extremal charged black hole in higher dimensions happen to belong to this class of exactly solvable models.

## V. HIGHER DIMENSIONAL MAJUMDAR-PAPAPETROU SPACETIMES

In the case of the higher dimensional Majumdar-Papapetrou spacetimes, [3]

$$\begin{aligned} ds^2 &= -U^{-2} dt^2 + U^{2/n} \delta_{ab} dx^a dx^b, \\ A_\mu^{\text{MP}} &= \sqrt{\frac{n+1}{2n}} U^{-1} \delta_\mu^0. \end{aligned} \quad (5.1)$$

the transformations Eq.(3.1) with

$$\Omega = U^{1/n} \quad (5.2)$$

lead to the metric  $\bar{g}_{\mu\nu}$  with

$$\bar{\alpha} = U^{-2}, \quad \bar{g}_{ab} = \delta_{ab}. \quad (5.3)$$

Therefore the potential for the field  $\bar{\psi}$  in the reference background metric  $\bar{g}_{\mu\nu}$  becomes

$$\bar{V} = -U^{-1} \Delta U \quad (5.4)$$

where

$$\Delta = \delta^{ab} \partial_a \partial_b \quad (5.5)$$

is the Laplace operator corresponding to the flat spatial metric  $\bar{g}_{ab} = \delta_{ab}$ .

The function  $U$  describing the Majumdar-Papapetrou metrics Eq.(5.1) satisfies the equation

$$\Delta U = -\frac{4\pi^{1+\frac{n}{2}}}{\Gamma(\frac{n}{2})} \sum_k M_k \delta^{n+2}(\mathbf{x} - \mathbf{x}_k). \quad (5.6)$$

The explicit solution of the Eq.(5.6) reads

$$U = 1 + \sum_k \frac{M_k}{\rho_k^n}, \quad \rho_k = |\mathbf{x} - \mathbf{x}_k|. \quad (5.7)$$

Here  $\mathbf{x} \equiv x^a$  and  $|\mathbf{x}|^2 \equiv \delta_{ab} x^a x^b$ . The index  $k = (1, \dots, N)$  enumerates the extremal black holes and  $x_k^a$  marks the spatial position of the  $k$ -th black hole.

Thus for the Majumdar-Papapetrou metrics, the potential  $\bar{V}$  vanishes everywhere, except the location of the horizons, where  $U \rightarrow \infty$ . On the horizon,  $\bar{V}$  formally vanishes but, because the field  $\psi$  may diverge there, one has to be quite accurate in dealing with distributions [4]. In fact, the  $\delta$ -functions do contribute to the potential  $\bar{A}_0$  and the self-energy of electric charges.

The solution for the static Green function has been obtained in [4]

$$\mathcal{G}_{00}(x, x') = -\frac{\Gamma(\frac{n}{2})}{4\pi^{1+\frac{n}{2}}} \frac{1}{U(x)U(x')} \left[ \frac{1}{R^n} + \sum_k \frac{M_k}{\rho_k^n \rho_k'^n} \right], \quad (5.8)$$

$$R = |\mathbf{x} - \mathbf{x}'|. \quad (5.9)$$

The Green function  $\mathcal{G}$  for the scalar field  $\psi$  obeying Eq.(2.19) is

$$\mathcal{G}(x, x') = \frac{\Gamma\left(\frac{n}{2}\right)}{4\pi^{1+\frac{n}{2}}\sqrt{UU'}} \left[ \frac{1}{R^n} + \sum_k \frac{M_k}{\rho_k^n \rho'_k{}^n} \right]. \quad (5.10)$$

In any reference spacetime with the metric  $\bar{g}_{\alpha\beta}$  described by Eq.(3.1) the corresponding Green function

$$\bar{\mathcal{G}}(x, x') = \mathcal{G}(x, x') \sqrt{U(x)U(x')}. \quad (5.11)$$

In the case of Majumdar-Papapetrou metrics the reference spacetime can be chosen such that  $\bar{g}_{ab}$  is a flat metric. In this case the vacuum divergent part of the ‘reference’ Green function is evidently described by the first term in brackets in Eq.(5.10)

$$\bar{\mathcal{G}}_{\text{div}}(x, x') = \frac{\Gamma\left(\frac{n}{2}\right)}{4\pi^{1+\frac{n}{2}}} \frac{1}{R^n}. \quad (5.12)$$

Thus, the exact formula for the regularized Green function in higher dimensional Majumdar-Papapetrou spacetimes is

$$\bar{\mathcal{G}}_{\text{reg}}(x, x') = \frac{\Gamma\left(\frac{n}{2}\right)}{4\pi^{1+\frac{n}{2}}} \sum_k \frac{M_k}{\rho_k^n \rho'_k{}^n}. \quad (5.13)$$

Substitution of this expression to Eq.(4.14) gives

$$\Delta m = e^2 \frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}} U^{-1} \sum_k \frac{M_k}{\rho_k^{2n}} + 2\pi e^2 \mathcal{B}, \quad (5.14)$$

In even-dimensional spacetimes, the anomaly  $\mathcal{B} = 0$ . In the case of a charge near a single extremal Reissner-Nordström black hole of mass  $M = Q$  in four dimensions, the Eq.(5.14) reproduces a well known result [5, 6]

$$E_{\text{ren}} = e^2 \frac{M}{2(\rho + M)^2} = e^2 \frac{M}{2r^2}, \quad (5.15)$$

where  $r = \rho + M$  is the Schwarzschild radial coordinate.

In odd dimensions the anomaly contribution enters the answer. In five dimensions we can use the result of our previous paper [1], where we have derived that for arbitrary static spacetimes

$$\begin{aligned} \mathcal{A}(x) &= \frac{1}{288\pi^2} \mathcal{R} - \frac{1}{64\pi^2} \ln(g) a_1(x), \\ a_1(x) &= \frac{1}{6} \mathcal{R} + V. \end{aligned} \quad (5.16)$$

In our particular case of the Majumdar-Papapetrou spacetimes, the DeWitt coefficient  $a_1 = 0$  and applying Eq.(4.10) one gets

$$\mathcal{B} = \frac{1}{288\pi^2} \mathcal{R}, \quad (5.17)$$

where

$$\mathcal{R} = \frac{3}{2} U^{-3} \delta^{ab} \partial_a U \partial_b U - 3U^{-2} \Delta U \quad (5.18)$$

is the scalar curvature of the four-dimensional spatial slice  $t = \text{const}$  which is described by the metric  $g_{ab} = U \delta_{ab}$ . Note that on the Majumdar-Papapetrou spacetimes, the term  $U^{-2} \Delta U$  vanishes both outside and at the horizons.

Thus, for  $D = 5$  we get  $\Delta m = U E_{\text{ren}}$  and

$$E_{\text{ren}} = \frac{e^2}{2\pi} \frac{1}{U^2} \sum_k \frac{M_k}{\rho_k^4} + \frac{e^2}{144\pi} \frac{1}{U} \mathcal{R}, \quad (5.19)$$

where  $U$  is given by Eq.(5.7) for  $n = 2$ . For a single extremal Reissner-Nordström black hole the self-energy is

$$E_{\text{ren}} = \frac{e^2}{2\pi} \frac{M}{(\rho^2 + M)^2} + \frac{e^2}{24\pi} \frac{\rho^2 M^2}{(\rho^2 + M)^4}, \quad (5.20)$$

In terms of the Schwarzschild radial coordinate  $r$ , which is given by  $r^2 = \rho^2 + M$ , the self-energy of the electric charge  $e$  near the five-dimensional extremal Reissner-Nordström black hole has the form

$$E_{\text{ren}} = \frac{e^2}{2\pi} \frac{M}{r^4} + \frac{e^2}{24\pi} \frac{(r^2 - M) M^2}{r^8}. \quad (5.21)$$

## VI. CONCLUSIONS

In this paper we demonstrated that renormalized shift of the self-mass for a pointlike electric charge in a static odd dimensional spacetime contains an anomaly contribution. This is a result of the gauge invariance of the corresponding  $(D-1)$ -dimensional Euclidean theory, which is broken by a covariant regularization. There exists a certain similarity of this property of the electromagnetic self-energy and the property of the self-energy for a scalar massless field. However, the corresponding form of gauge transformations preserving the form of the energy action is different for these two cases. It is quite interesting that a solution of the rather old classical problem of the self-energy in both cases can be reduced to the problem of Euclidean quantum field theory in the space of the codimension one. This makes it possible to use well developed tools of quantum theory to perform the classical calculations. In particular, the results of [7] allow one to expect that the renormalized value of the mass-shift and the calculated anomalies do not depend of the details of the regularization scheme. Speaking in terms of the original  $D$ -dimensional classical theory, the above described gauge transformations relate spacetimes with nontrivial transformation of the  $D$ -dimensional metric. In special cases one can use this transformation in order to simplify the problem. We demonstrated that for the wide class of Majumdar-Papapetrou spacetimes this allows one to obtain an explicit answer. One can also try to apply the gauge induced transformation in order to find a spacetime where appropriate approximation can be used. Using the known anomaly one can obtain a corresponding approximation for the mass-shift in the original, physically interesting case. An interesting question is whether

the developed approach can be generalized for the calculation of the gravitational self-energy in static spacetimes. Another interesting question is: whether it is possible to generalize this approach to stationary spacetimes. In more general terms, it is interesting whether quantum tools can be used for solving other classical problems, for example, calculation of the self-energy for special distributed configuration of charges and/or self-energy of dipole (multipole) charge configurations.

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  - [8] Note that here we use Gaussian units. For calculation of the vacuum fluctuations in quantum field theory it

is more traditional to use Heaviside-Lorentz units which differ from the Gaussian units by numerical factors. In Heaviside-Lorentz units the first term in the Maxwell action Eq.(2.2) has the coefficient  $1/4$  instead of  $1/16\pi$ . The substitution  $A_\mu \rightarrow A_\mu\sqrt{4\pi}$  and  $J^\mu \rightarrow J^\mu/\sqrt{4\pi}$  transforms Gaussian form of the equations to the Heaviside-Lorentz one. As the consequence  $e^2 \rightarrow e^2/(4\pi)$  and the self-energy in Heaviside-Lorentz units acquires an additional  $1/(4\pi)$  factor. The mass shift of a scalar charge  $\Delta m = q^2\mathcal{R}/576\pi^2$ , which is given by Eq.(5.13) of the paper [2], corresponds to Heaviside-Lorentz units. In Gaussian units it reads  $\Delta m = q^2\mathcal{R}/144\pi$ .